Homogeneous first-order differential equations

Find the solution of the following second-order homogeneous differential equations:

1. $y'' - y' - 2y = 0$
2. $y'' - 4y' + 3y = 0$
3. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$
4. $y'' + 16y = 0$
5. $y'' + 2y' + 2y = 0$
6. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$
7. $y'' + 6y' + 9y = 0$

Solution

1. $$y'' - y' - 2y = 0$$

   We start by finding its characteristic equation. By substituting y'' with r^2, y' with r, and y with 1, the characteristic equation becomes:

   $$r^2 - r - 2 = 0$$

   Next, we solve the quadratic equation for r:

   $$r^2 - r - 2 = 0$$

   We factorize the quadratic equation:

   $$(r - 2)(r + 1) = 0$$

   By setting each factor equal to zero, we get the roots:

   1. $r - 2 = 0 \quad \Rightarrow \quad r = 2$ 2. $r + 1 = 0 \quad \Rightarrow \quad r = -1$

   Since the roots are real and distinct, the general solution of the differential equation is:

   $$y(x) = C_1 e^{2x} + C_2 e^{-x}$$

   where C_1 and C_2 are arbitrary constants determined by initial conditions.

2. $$y'' - 4y' + 3y = 0$$

   We start by finding its characteristic equation. By substituting y'' with r^2, y' with r, and y with 1, we get:

   $$r^2 - 4r + 3 = 0$$

   We solve the quadratic equation for r:

   $$r^2 - 4r + 3 = 0$$

   We factorize the equation:

   $$(r - 1)(r - 3) = 0$$

   By setting each factor equal to zero, we get the roots:

   1. $r - 1 = 0 \quad \Rightarrow \quad r = 1$ 2. $r - 3 = 0 \quad \Rightarrow \quad r = 3$

   Since the roots are real and distinct, the general solution of the differential equation is:

   $$y(x) = C_1 e^{x} + C_2 e^{3x}$$

   where C_1 and C_2 are arbitrary constants determined by initial conditions.

3.

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

We start by finding its characteristic equation. By substituting $\frac{d^2y}{dx^2}$ with r^2, $\frac{dy}{dx}$ with r, and y with 1, we obtain:

$$r^2 - 2r + 1 = 0$$

We solve the quadratic equation for r:

$$r^2 - 2r + 1 = 0$$

We notice that it is a perfect square:

$$(r-1)^2 = 0$$

By setting the factor equal to zero, we obtain the double root:

$$r - 1 = 0 \quad \Rightarrow \quad r = 1$$

Since we have a repeated real root, the general solution of the differential equation is:

$$y(x) = (C_1 + C_2 x)e^{rx}$$

Substituting $r = 1$:

$$y(x) = (C_1 + C_2 x)e^{x}$$

Therefore, the general solution is:

$$y(x) = (C_1 + C_2 x)e^{x}$$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

4.

$$y'' + 16y = 0$$

We start by finding its characteristic equation. By substituting y'' with r^2 and y with 1, we obtain:

$$r^2 + 16 = 0$$

We solve the quadratic equation for r:

$$r^2 = -16$$

$$r = \pm\sqrt{-16} = \pm 4i$$

Since the roots are complex conjugates of the form $r = \alpha \pm \beta i$, where $\alpha = 0$ and $\beta = 4$, the general solution of the differential equation is:

$$y(x) = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)$$

Substituting $\alpha = 0$ and $\beta = 4$:

$$y(x) = e^{0\cdot x}\left(C_1\cos(4x) + C_2\sin(4x)\right)$$

Simplifying $e^{0\cdot x} = 1$:

$$y(x) = C_1\cos(4x) + C_2\sin(4x)$$

Therefore, the general solution is:

$$y(x) = C_1\cos(4x) + C_2\sin(4x)$$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

5.

$$y'' + 2y' + 2y = 0$$

We start by finding its characteristic equation. By substituting y'' with r^2, y' with r, and y with 1, we obtain:

$$r^2 + 2r + 2 = 0$$

We solve the quadratic equation for r:

$$r^2 + 2r + 2 = 0$$

Using the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = 2$, and $c = 2$.

We calculate the discriminant:

$$\Delta = b^2 - 4ac = (2)^2 - 4(1)(2) = 4 - 8 = -4$$

The discriminant is negative, indicating that the roots are complex conjugates.

We calculate the roots:

$$r = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

Therefore, the roots are $r = -1 + i$ and $r = -1 - i$.

Since the roots are complex conjugates of the form $r = \alpha \pm \beta i$, where $\alpha = -1$ and $\beta = 1$, the general solution of the differential equation is:

$$y(x) = e^{\alpha x}\left(C_1\cos(\beta x) + C_2\sin(\beta x)\right)$$

Substituting $\alpha = -1$ and $\beta = 1$:

$$y(x) = e^{-x}\left(C_1\cos(x) + C_2\sin(x)\right)$$

Therefore, the general solution is:

$$y(x) = e^{-x}\left(C_1 \cos(x) + C_2 \sin(x)\right)$$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

6.
$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$$

We start by finding its characteristic equation. By substituting $\frac{d^2y}{dx^2}$ with r^2, $\frac{dy}{dx}$ with r, and y with 1, we obtain:

$$r^2 - 2r + 5 = 0$$

We solve the quadratic equation for r:

$$r^2 - 2r + 5 = 0$$

Using the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = -2$, and $c = 5$.

We calculate the discriminant:

$$\Delta = b^2 - 4ac = (-2)^2 - 4(1)(5) = 4 - 20 = -16$$

The discriminant is negative, indicating that the roots are complex conjugates.

We calculate the roots:

$$r = \frac{-(-2) \pm \sqrt{-16}}{2 \cdot 1} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Therefore, the roots are $r = 1 + 2i$ and $r = 1 - 2i$.

Since the roots are of the form $r = \alpha \pm \beta i$, where $\alpha = 1$ and $\beta = 2$, the general solution of the differential equation is:

$$y(x) = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)$$

Substituting $\alpha = 1$ and $\beta = 2$:

$$y(x) = e^{x}\left(C_1 \cos(2x) + C_2 \sin(2x)\right)$$

Therefore, the general solution is:

$$y(x) = e^{x}\left(C_1 \cos(2x) + C_2 \sin(2x)\right)$$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

7.

$$y'' + 6y' + 9y = 0$$

We start by finding its characteristic equation. By substituting y'' with r^2, y' with r, and y with 1, we obtain:

$$r^2 + 6r + 9 = 0$$

We solve the quadratic equation for r:

$$r^2 + 6r + 9 = 0$$

We notice that it is a perfect square:

$$(r+3)^2 = 0$$

By setting the factor equal to zero, we obtain the double root:

$$r + 3 = 0 \quad \Rightarrow \quad r = -3$$

Since we have a repeated real root, the general solution of the differential equation is:

$$y(x) = (C_1 + C_2 x)e^{rx}$$

Substituting $r = -3$:

$$y(x) = (C_1 + C_2 x)e^{-3x}$$

Therefore, the general solution is:

$$y(x) = (C_1 + C_2 x)e^{-3x}$$

where C_1 and C_2 are arbitrary constants determined by initial conditions.